

# On the Sum of Linear Coefficients of a Boolean Valued Function

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ABSTRACT. Let  $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$  be a Boolean valued function having total degree  $d$ . Then a conjecture due to Servedio and Gopalan asserts that  $\sum_{i=1}^n \widehat{f}(i) \leq \sum_{j=1}^d \widehat{\text{Maj}}_d(j)$  where  $\text{Maj}_d$  is the majority function on  $d$  bits. Here we give some alternative formalisms of this conjecture involving the discrete derivative operators on  $f$ .

## 1. Introduction: Fourier Analysis of Boolean Functions

We are concerned with *Boolean-valued* functions  $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$  which form a subset of *Boolean* functions  $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ . Every Boolean function  $f : \{-1, 1\}^n \rightarrow \mathbb{R}$  has a unique *Fourier expansion* given by

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i,$$

where the real numbers  $\widehat{f}(S)$  are the *Fourier coefficients* of  $f$  given by the formula

$$\widehat{f}(S) = \mathbf{E} \left[ f(x) \prod_{i \in S} x_i \right].$$

(Here and everywhere else in the paper, the expectation  $\mathbf{E}[\cdot]$  is with respect to the uniform probability distribution on  $\{-1, 1\}^n$ .) The *Parseval's identity* is the fact that  $\sum_{S \subseteq [n]} \widehat{f}(S)^2 = \mathbf{E}[f(x)^2]$ . In particular, if  $f$  is boolean-valued then this implies that  $\sum_{S \subseteq [n]} \widehat{f}(S)^2 = 1$ .

Given  $f : \{-1, 1\}^n \rightarrow \mathbb{R}$  and  $i \in [n]$ , we define the *discrete derivative*  $\partial_i f : \{-1, 1\}^n \rightarrow \mathbb{R}$  by

$$\partial_i f(x) = \frac{f(x_1, x_2, \dots, 1, x_{i+1}, \dots, x_n) - f(x_1, x_2, \dots, -1, x_{i+1}, \dots, x_n)}{2}.$$

The *influence* of the  $i$ th coordinate on  $f$  is defined by

$$\mathbf{Inf}_i[f] = \mathbf{E}[(\partial_i f)^2] = \sum_{S \ni i} \widehat{f}(S)^2.$$

In the particular case when  $f$  is Boolean-valued, the derivative  $\partial_i f$  is  $\{-1, 0, 1\}$ -valued. The *total influence* of  $f$  is

$$\mathbf{Inf}[f] = \sum_{i=1}^n \mathbf{Inf}_i[f] = \sum_{S \subseteq [n]} |S| \cdot \widehat{f}(S)^2.$$

The *total degree* of  $f$  is defined by

$$\deg(f) = \max\{|S| : \widehat{f}(S) \neq 0\}.$$

Note that for a Boolean-valued function  $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$

$$\mathbf{Inf}[f] = \sum_{S \subseteq [n]} |S| \cdot \widehat{f}(S)^2 \leq \deg(f) \cdot \sum_{S \subseteq [n]} \widehat{f}(S)^2 = \deg(f),$$

where we used the Parseval's identity to deduce the last step.

The linear coefficients of  $f$  are the  $n$  coefficients  $\widehat{f}(\{1\}), \widehat{f}(\{2\}), \dots, \widehat{f}(\{n\})$ , and hereon we omit writing the curly braces inside to denote them. We are concerned with the following conjecture here:

**CONJECTURE 1.1** (Gopalan-Servedio [1]). Let  $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$  have total degree  $d$ . Let  $\text{Maj}_d : \{-1, 1\}^d \rightarrow \{-1, 1\}$  be defined by  $\text{Maj}_d(x) = \text{sgn}(x_1 + x_2 + \dots + x_d)$  with  $\text{sgn}(0) = -1$ . Then

$$\sum_{i=1}^n \widehat{f}(i) \leq \sum_{j=1}^d \widehat{\text{Maj}}_d(j).$$

**REMARK 1.2.** The conjecture is trivial when  $\deg(f) = n$  since

$$\sum_{i=1}^n \widehat{f}(i) = 2^{-n} \sum_{x \in \{-1, 1\}^n} f(x) \cdot (x_1 + x_2 + \dots + x_n) \leq 2^{-n} \sum_{x \in \{-1, 1\}^n} |x_1 + x_2 + \dots + x_n| = \sum_{j=1}^n \widehat{\text{Maj}}_n(j).$$

## 2. Alternative Formalisms of the conjecture

**THEOREM 2.1.** *The Conjecture 1.1 is equivalent to each of the following inequalities*

$$\begin{aligned} \mathbf{Inf}[f] - \mathbf{Inf}[\text{Maj}_d] &\leq 2 \cdot \sum_{k=1}^n \mathbf{Pr}[\partial_k f = -1], \\ \sum_{k=1}^n (\mathbf{Pr}[\partial_k f = 1] - \mathbf{Pr}[\partial_k \text{Maj}_d = 1]) &\leq \sum_{j=1}^n \mathbf{Pr}[\partial_j f = -1], \\ 2 \cdot \sum_{k=1}^n \mathbf{Pr}[\partial_k f = 1] &\leq \mathbf{Inf}[f] + \mathbf{Inf}[\text{Maj}_d]. \end{aligned}$$

**PROOF.** Let

$$f(x) = \sum_{S \subseteq [n]} \widehat{f}(S) \prod_{i \in S} x_i,$$

and

$$\partial_i f(y) := \frac{f(y_1, y_2, \dots, 1, y_{i+1}, \dots, y_n) - f(y_1, y_2, \dots, -1, y_{i+1}, \dots, y_n)}{2}$$

for all  $y \in \{-1, 1\}^{n-1}$  which takes values  $+1, 0, -1$ . We have

$$\begin{aligned} \mathbf{Pr}[\partial_i f = 0] &= 2^{-(n-1)} \sum_{y \in \{-1, 1\}^{n-1}} (1 + \partial_i f(y)) \cdot (1 - \partial_i f(y)) = 1 - \mathbf{Inf}_i[f], \\ \mathbf{Pr}[\partial_i f = 1] &= 2^{-(n-1)} \sum_{y \in \{-1, 1\}^{n-1}} \frac{(\partial_i f(y) + 1) \cdot \partial_i f(y)}{2} = \frac{1}{2}(\mathbf{Inf}_i[f] + \widehat{f}(i)), \\ \mathbf{Pr}[\partial_i f = -1] &= 2^{-(n-1)} \sum_{y \in \{-1, 1\}^{n-1}} \frac{(\partial_i f(y) - 1) \cdot \partial_i f(y)}{2} = \frac{1}{2}(\mathbf{Inf}_i[f] - \widehat{f}(i)). \end{aligned}$$

where we used the fact that  $\widehat{f}(i) = \mathbf{E}[(\partial_i f)]$ . Each of the assertions follow easily from the above equations.  $\square$

## References

1. Ryan O'Donnell, *Open problems in analysis of boolean functions*, arXiv preprint arXiv:1204.6447 (2012).

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